ANNEX

F1 :
$$C = \nabla\Phi^t \nabla\Phi.$$

F2 :
$$\Phi(\mathbf{X}) = \sum_{j=0}^{2} \Lambda_j(\mathbf{X})\mathbf{Q}_j \implies \nabla\Phi = \sum_{j=0}^{2} \alpha_j \otimes \mathbf{Q}_j \text{ where } \alpha_j = \frac{l_j}{2A}\mathrm{n}_j.$$

F3 :
$$C = \sum_{j,k} (\mathbf{Q}_j \,.\, \mathbf{Q}_k)\,(\alpha_j \otimes \alpha_k)\,.$$

F4 :
$$\begin{aligned} C &= \sum_j r^2\,(\alpha_j \otimes \alpha_j) + \sum_{\substack{j,k \\ j\neq k}} \left(r^2 - \frac{L_p^2}{2}\right)(\alpha_j \otimes \alpha_k) \\ &= r^2 \sum_{j,k} (\alpha_j \otimes \alpha_k) - \frac{1}{2} \sum_{\substack{j,k \\ j\neq k}} L_p^2 (\alpha_j \otimes \alpha_k) \\ &= -\frac{1}{2} \sum_{\substack{j,k \\ j\neq k}} L_p^2 (\alpha_j \otimes \alpha_k) \end{aligned}$$

F5 :
$$\begin{aligned} W &= \frac{\lambda}{2}(tr\,E)^2 + \mu tr\,E^2 \\ tr\,E &= \frac{1}{2}\boxed{tr\,C} - 1 \\ tr\,E^2 &= \frac{1}{4}\boxed{tr\,C^2} - tr\,E - \frac{1}{2} \end{aligned}$$

F6 :
$$tr\,C = -\frac{1}{2} \sum_{\substack{j,k \\ j\neq k}} L_p^2\,(\alpha_j \,.\, \alpha_k)\,.$$

F7 :
$$\overrightarrow{\mathbf{P}_p\mathbf{P}_q} \,.\, \overrightarrow{\mathbf{P}_p\mathbf{P}_r} = \begin{cases} l_j\, l_k\, cos(\theta_p) = \frac{1}{2}\left(l_j^2 + l_k^2 - l_p^2\right) = K_p & \text{if } q \neq r\,,\ j = comp(p,q) \\ & \text{and } k = comp(p,r) \\ l_j^2 & \text{if } q = r\,,\ j = comp(p,q) \end{cases}$$

R1 :
$$(\alpha_j \,.\, \alpha_k) = \begin{cases} -\frac{l_j\, l_k}{4A^2}\, cos(\theta_p) = -\frac{1}{4A^2} K_p & \text{if } j \neq k \text{ and } p = comp(j,k)\,, \\ \frac{l_j^2}{4A^2} & \text{if } j = k \end{cases}$$

F8 :

$$tr\ C = \frac{1}{4\mathcal{A}^2} \sum_p L_p^2\, \mathcal{K}_p\,.$$

F9 :

$$\boxed{tr\ E = \frac{1}{8\mathcal{A}^2} \sum_p \left(L_p^2 - l_p^2\right) \mathcal{K}_p = \frac{1}{8\mathcal{A}^2} \left[\left(L_0^2 - l_0^2\right) \mathcal{K}_0 + \left(L_1^2 - l_1^2\right) \mathcal{K}_1 + \left(L_2^2 - l_2^2\right) \mathcal{K}_2\right]}$$

F10 :

$$tr\ C^2 = \frac{1}{4} \sum_{\substack{j,k\\ j\neq k}} \sum_{\substack{m,n\\ m\neq n}} L_p^2\, L_q^2 \left(\alpha_j \,.\, \alpha_n\right)\left(\alpha_k \,.\, \alpha_m\right)$$

F11 :

$$\begin{aligned}
\mathbf{F}_0^T = \frac{\partial W}{\partial \mathbf{Q}_0} \;=\; & \frac{1}{32\mathcal{A}^4}\Bigg[(\lambda+\mu)\left(\sum_p \left(L_p^2 - l_p^2\right) \mathcal{K}_p\right)\left(\overrightarrow{\mathbf{Q}_2\mathbf{Q}_0}\, \mathcal{K}_1 + \overrightarrow{\mathbf{Q}_1\mathbf{Q}_0}\, \mathcal{K}_2\right) \\
& + \mu\Bigg(\left(L_1^2 - l_1^2\right) l_0^2\left(l_2^2\, \overrightarrow{\mathbf{Q}_2\mathbf{Q}_0} - \mathcal{K}_0 \overrightarrow{\mathbf{Q}_1\mathbf{Q}_0}\right) + \left(L_2^2 - l_2^2\right) l_0^2\left(l_1^2\, \overrightarrow{\mathbf{Q}_1\mathbf{Q}_0} - \mathcal{K}_0 \overrightarrow{\mathbf{Q}_2\mathbf{Q}_0}\right) \\
& - \left(L_0^2 - l_0^2\right)\left(l_1^2\, \mathcal{K}_1\, \overrightarrow{\mathbf{Q}_1\mathbf{Q}_0} + l_2^2\, \mathcal{K}_2\, \overrightarrow{\mathbf{Q}_2\mathbf{Q}_0}\right)\Bigg)\Bigg]
\end{aligned}$$

F12 :

$$\nabla\Phi \;=\; \sum_{j=0}^{3} \alpha_j \otimes \mathbf{Q}_j \qquad\qquad \alpha_j = \frac{\mathcal{A}_j}{3\mathcal{V}}\mathbf{n}_j$$

F13 :

$$C \;=\; \sum_{j,k} \left(\mathbf{Q}_j \,.\, \mathbf{Q}_k\right)\left(\alpha_j \otimes \alpha_k\right)$$

F14 :

$$\left(\mathbf{Q}_j \,.\, \mathbf{Q}_k\right) = \begin{cases} r^2 - \dfrac{L_{jk}^2}{2} & \text{if } j \neq k \\ r^2 & \text{if } j = k \end{cases} \qquad \Longrightarrow \qquad \boxed{C = -\frac{1}{2} \sum_{\substack{j,k\\ j\neq k}} L_p^2 \left(\alpha_j \otimes \alpha_k\right)}$$

F15 :

$$\begin{aligned}
W \;&=\; \frac{\lambda}{2}\left(tr\ E\right)^2 + \mu tr\ E^2 \\
tr\ E \;&=\; \frac{1}{2}\boxed{tr\ C} - \frac{3}{2} \\
tr\ E^2 \;&=\; \frac{1}{4}\boxed{tr\ C^2} - tr\ E - \frac{3}{4}
\end{aligned}$$

F16 :

$$(\alpha_j \cdot \alpha_k) = \begin{cases} -\dfrac{1}{9\mathcal{V}^2} \mathcal{K}_{jk} & \text{if } j \neq k \\ \dfrac{1}{9\mathcal{V}^2} \mathcal{A}_j^2 & \text{if } j = k \end{cases}$$

$$\text{with } \mathcal{K}_{jk} = \frac{2\, l_{lm}^2 \left(l_{jl}^2 + l_{kl}^2 - l_{jk}^2\right) - \left(l_{jl}^2 + l_{lm}^2 - l_{jm}^2\right)\left(l_{kl}^2 + l_{lm}^2 - l_{km}^2\right)}{16}$$

F17 :

$$tr\, C = \frac{1}{9\mathcal{V}^2} \sum_{\text{Edges } (j,k)} L_{jk}^2\, \mathcal{K}_{jk}$$

F18 :

$$tr\, C^2 = \frac{1}{81\, \mathcal{V}^4} \Bigg[\frac{1}{2} \sum_{\text{edges}(j,k)} \underline{L_{jk}^4}\, \left(\underline{\mathcal{K}_{jk}^2} + \mathcal{A}_j^2\, \mathcal{A}_k^2\right) \qquad \text{6 terms}$$

$$+ \sum_{\substack{\text{pairs of edges} \\ ((j,k),(j,m))}} \underline{L_{jk}^2\, L_{jm}^2}\, \left(\underline{\mathcal{K}_{jk}\, \mathcal{K}_{jm}} - \mathcal{A}_j^2\, \mathcal{K}_{km}\right) \qquad \text{12 terms}$$

$$+ \sum_{\substack{\text{pairs of edges} \\ ((j,k),(m,n))}} L_{jk}^2\, L_{mn}^2\, \left(\mathcal{K}_{jn}\, \mathcal{K}_{km} + \mathcal{K}_{jm}\, \mathcal{K}_{kn}\right) \Bigg] \qquad \text{3 terms}$$

F19 :

$$\boxed{W = \frac{1}{324\, \mathcal{V}^4} \left\{ \left(\frac{\lambda + \mu}{2} \right) \left[\sum_{\text{Edges}(j,k)} \left(L_{jk}^2 - l_{jk}^2\right) \mathcal{K}_{jk} \right]^2 + \mu\, \tilde{\Delta} \right\}}$$

F20 :

$$\tilde{\Delta} = \frac{1}{2} \sum_{\text{edges } (j,k)} \left(L_{jk}^2 - l_{jk}^2\right)^2 \mathcal{A}_j^2\, \mathcal{A}_k^2$$

$$- \sum_{\substack{\text{pairs of edges} \\ ((j,k),(j,m))}} \left(L_{jk}^2 - l_{jk}^2\right)\left(L_{jm}^2 - l_{jm}^2\right)\, \mathcal{A}_j^2\, \mathcal{K}_{km}$$

$$+ \sum_{\substack{\text{pairs of edges} \\ ((j,k),(m,n))}} \left(L_{jk}^2 - l_{jk}^2\right)\left(L_{mn}^2 - l_{mn}^2\right)\, \left(\mathcal{K}_{jn}\, \mathcal{K}_{km} + \mathcal{K}_{jm}\, \mathcal{K}_{kn} - \mathcal{K}_{jk}\, \mathcal{K}_{mn}\right)$$

F21 :

$$\mathbf{F}_p = \frac{1}{162\,\mathcal{V}^4}\Bigg\{(\lambda+\mu)\left[\sum_{\text{Edges }(j,k)} \left(L_{jk}^2 - l_{jk}^2\right) \mathcal{K}_{jk}\right]\left(\sum_{\text{Edges}(j,p)} \overrightarrow{\mathbf{Q}_j\mathbf{Q}_p}\,\mathcal{K}_{jp}\right)$$
$$+\mu\Bigg[\sum_{\text{Edges}(j,p)} \left(L_{jp}^2 - l_{jp}^2\right) \mathcal{A}_p^2\left(\overrightarrow{\mathbf{Q}_j\mathbf{Q}_p}\,\mathcal{A}_j^2 - \overrightarrow{\mathbf{Q}_k\mathbf{Q}_p}\,\mathcal{K}_{jk} - \overrightarrow{\mathbf{Q}_m\mathbf{Q}_p}\,\mathcal{K}_{jm}\right)$$
$$+\sum_{\substack{\text{Edges }(j,k)\\ j,k\neq p}} \left(L_{jk}^2 - l_{jk}^2\right)\left(\overrightarrow{\mathbf{Q}_m\mathbf{Q}_p}\,\mathcal{K}_{jkmp} - \overrightarrow{\mathbf{Q}_j\mathbf{Q}_p}\,\mathcal{A}_j^2\mathcal{K}_{kp} - \overrightarrow{\mathbf{Q}_k\mathbf{Q}_p}\,\mathcal{A}_k^2\mathcal{K}_{jp}\right)\Bigg]\Bigg\}$$

R2 :

$$W = \frac{1}{128\mathcal{A}^4}\Bigg[(\lambda+\mu)\left(\sum_p \left(L_p^2 - l_p^2\right)\mathcal{K}_p\right)^2$$
$$+\mu\left(\sum_p \left(L_p^2 - l_p^2\right)^2\, l_j^2 l_k^2 - 2\sum_{\substack{j,k\\ j\neq k}} \left(L_j^2 - l_j^2\right)\left(L_k^2 - l_k^2\right)\, l_p^2\mathcal{K}_p\right)\Bigg]$$

$$p = comp(j,k)$$

E1 :

$$m_i \frac{d^2\mathbf{P}_i}{dt^2} = \gamma_i \frac{d\mathbf{P}_i}{dt} + \mathbf{F}_i$$

E2 :

$$M\ddot{\mathbf{U}} + C\dot{\mathbf{U}} + K\mathbf{U} = \mathbf{R}$$

E3 :

$$\left(\frac{m_i}{\Delta t^2} - \frac{\gamma_i}{2\Delta t}\right)P_i^{t+1} = F_i + \frac{2m_i}{\Delta t^2}P_i^t - \left(\frac{m_i}{\Delta t^2} + \frac{\gamma_i}{2\Delta t}\right)P_i^{t-1}$$